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Spherical Functions on the Symmetric Groups

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1. BACKGROUND AND PRELIMINARIES

Let H be a subgroup of a finite group G. Denote by $\mathbb{C}[G]^H$ the centralizer of H in the complex group algebra $\mathbb{C}[G]$. Then $\mathbb{C}[G]^H$ is easily seen to be the set of all elements whose coefficients are constant on H-conjugacy classes of G; that is, $\mathbb{C}[G]^H$ may be thought of as the space of H-class functions on G. Travis [4] has studied the representation theory of $\mathbb{C}[G]^H$. We summarize here some of his results.

Let $\hat{G}$ be the set of complex irreducible characters of G and $\hat{H}$ those of H. Let $\chi \in \hat{G}$, $\psi \in \hat{H}$, and let $c_{\chi\psi}$ be the multiplicity with which ψ is contained in the restriction of χ to H. Then Travis defines the spherical function $Y_{\chi\psi}$ on G by

$$Y_{\chi\psi}(\tau) = (1/\|H\|) \sum_{\sigma \in H} \chi(\tau\sigma)\, \psi(\sigma^{-1}) \qquad (\tau \in G). \tag{1}$$

Then $Y_{\chi\psi}$ is an H-class function on G and will be a nonzero function if and only if $c_{\chi\psi} \neq 0$. If $\tilde{\psi}$ denotes the function equal to $(G:H)\psi$ on H and 0 off H, then $Y_{\chi\psi} = \chi * \tilde{\psi}$, where $*$ is convolution in $\mathbb{C}[G]$. It follows that

$$Y_{\chi\psi} * \chi' = \delta_{\chi\chi'}(Y_{\chi\psi}/f_\chi), \tag{2}$$

$$Y_{\chi\psi} * \tilde{\psi}' = \delta_{\psi\psi'}(Y_{\chi\psi}/f_\psi), \tag{3}$$

and

$$Y_{\chi\psi} * Y_{\chi'\psi'} = \delta_{\chi\chi'}\delta_{\psi\psi'}(Y_{\chi\psi}/f_\chi f_\psi), \tag{4}$$

where the δ's are Kronecker δ's and f_χ and f_ψ are the degrees of χ and ψ, respectively.

Moreover,

$$\chi = \sum_{\psi} f_\psi Y_{\chi\psi} \tag{5}$$

and

$$\tilde{\psi} = \sum_{\chi} f_\chi Y_{\chi\psi} . \tag{6}$$

The spherical functions are in the center of $\mathbb{C}[G]^H$, and from (4) it follows that they are orthogonal, hence linearly independent, and up to the scalar factors $f_\chi f_\psi$, idempotent. Travis shows that they actually span the center of $\mathbb{C}[G]^H$ and that the $(f_\chi f_\psi / \| G \|)\, Y_{\chi\psi}$'s are the minimal central idempotents of $\mathbb{C}[G]^H$, which is semisimple.

Moreover, the nonzero spherical functions are in one-to-one correspondence with the irreducible representations of $\mathbb{C}[G]^H$, and may be interpreted as the characters of these representations. If Q is such a representation, and if L is an H-class sum in G, so that $L \in \mathbb{C}[G]^H$, and if $Y_{\chi\psi}$ corresponds to Q, then

$$\| L \| \, Y_{\chi\psi}(L) = \operatorname{tr} Q(L).$$

If all restriction multiplicities from G to H are 0 or 1, then H is called multiplicity-free. In that case $\mathbb{C}[G]^H$ is commutative, all its representations are one-dimensional, and the spherical functions span all of $\mathbb{C}[G]^H$. In this paper we construct the spherical functions (by finding the central idempotents) when $G = S_n$, the symmetric group on the symbols $1,\ldots, n$, and $H = S_{n-1}$. The work also leads to the averaging theorem, a relation between the spherical functions for $H \subset G$ and the characters of G, which holds for all finite groups.

2. The S_{n-1}-Conjugacy Classes of S_n

Let S_{n-1} be embedded in S_n as the group of all permutations on $1,\ldots, n$ that fix the symbol 1. As is well known, every element $\tau \in S_n$ can be written as a product of disjoint cycles $\tau = (a_1 \cdots a_{\lambda_1})(b_1 \cdots b_{\lambda_2}) \cdots (d_1 \cdots d_{\lambda_r})$, unique up to a reordering of the cycles and cyclic rearrangement of the symbols within each cycle. Hence τ determines the partition $\lambda_1 + \cdots + \lambda_r = n$. If $\rho \in S_n$ then we have

$$\rho\tau\rho^{-1} = (\rho(a_1) \cdots \rho(a_{\lambda_1}))\,(\rho(b_1) \cdots \rho(b_{\lambda_2})) \cdots (\rho(d_1) \cdots \rho(d_{\lambda_r})),$$

and so two elements of S_n are conjugate if and only if their cycle structures define the same partition of n. Hence the set of all elements of S_n having cycle structure $\lambda = (\lambda_1, \ldots, \lambda_r)$ form a conjugacy class of S_n, denoted (λ), and the classes of S_n are thus in one-to-one correspondence with the partitions of n.

It is immediate from the above that two elements of S_n will be conjugate by an element of S_{n-1} if and only if they have the same cycle structure and if they contain the symbol 1 in cycles of equal length.

DEFINITION. A *1-partition of n* is a partition of n with one part distinguished. If a 1-partition has parts $\lambda_1, ..., \lambda_r$ with $\lambda_1 + \cdots + \lambda_r = n$, and if λ_i is the distinguished part, we denote it by $\underline{\lambda}$ or $(\lambda_1 \cdots \underline{\lambda}_i \cdots \lambda_r)$.

Thus every element of S_n determines a 1-partition. The lengths of the cycles yield the parts of the 1-partition, and the distinguished part is the length of the cycle containing 1. If we denote by $(\underline{\lambda})$ the set of all elements of S_n yielding the 1-partition $\underline{\lambda}$, then we have that $(\underline{\lambda})$ is an S_{n-1}-conjugacy class of S_n, and that the S_{n-1}-classes of S_n are thus in one-to-one correspondence with the 1-partitions of n. Further, if the partition $\lambda = (\lambda_1, ..., \lambda_r)$ has s *distinct* sized parts, then there will be s different 1-partitions denoted $\underline{\lambda}_1, ..., \underline{\lambda}_s$ arising from λ, where in each we distinguish a different sized part of λ. It is clear that the S_n-class (λ) is the disjoint union of the S_{n-1}-classes $(\underline{\lambda}_1), ..., (\underline{\lambda}_s)$.

If $p(n)$ is the number of partitions of n, and $q(n)$ is the number of 1-partitions of n, then we have

PROPOSITION 1. $q(n) = p(n-1) + p(n-2) + \cdots + p(1) + 1$.

Proof. If a 1-partition has a distinguished part equal to m, then the remaining parts constitute a partition of $n - m$. Since the distinguished part may be anything from 1 to n, the result follows.

3. THE SPHERICAL FUNCTIONS FOR $S_{n-1} \subset S_n$

It will be convenient to have a method of indexing the nonzero spherical functions analogous to the indexing of the S_{n-1}-classes of S_n. First we label the characters of S_n and S_{n-1} by partitions of n and of $n - 1$, respectively, in the classical way, denoting an element of $\hat{S}_n$ by χ^λ, where λ is a partition of n. We will also use the notation $[\lambda]$ for the character χ^λ. (For details, see [3].) It is then well known that on restriction to S_{n-1},

$$\chi^\lambda = \sum \psi^\mu,$$

where the summation is over all partitions μ of $n - 1$ that can be obtained by decreasing a part of λ by 1, and each such μ appears exactly once.

In particular, S_{n-1} is a multiplicity-free subgroup of S_n. Moreover, a particular pair χ^λ, ψ^μ with $\psi^\mu \in \chi^\lambda |_{S_{n-1}}$ may be specified by giving the partition λ of n, and distinguishing the part of λ to be decreased by 1 to obtain μ. In other words, we may use the 1-partitions to indicate pairs χ, ψ with $\chi \in \hat{S}_n$, $\psi \in \hat{S}_{n-1}$ and with $c_{\chi\psi} \neq 0$.

Hence, since each nonzero spherical function corresponds to such a pair of characters, we will denote the spherical function $Y_{[\lambda_1 \cdots \lambda_r],[\lambda_1 \cdots (\lambda_i - 1) \cdots \lambda_r]}$ by $Y_{[\lambda_1 \cdots \underline{\lambda}_i \cdots \lambda_r]}$, or more simply by $[\underline{\lambda}] = [\lambda_1 \cdots \underline{\lambda}_i \cdots \lambda_r]$.

So the semisimple algebra $\mathbb{C}[S_n]^{S_{n-1}}$ is commutative, has dimension equal to $q(n)$, and is spanned by the idempotents $(f_\chi f_\psi / n!)\, Y_{\chi\psi}$. Since the central idempotents of a semisimple algebra are unique, to find the spherical functions it will suffice to find $q(n)$ mutually annihilating idempotents in $\mathbb{C}[S_n]^{S_{n-1}}$.

To achieve this, we use the Young technique for constructing the representations of S_n. We quote the necessary facts, the proofs of which may be found, e.g., in [1].

Given a partition $\lambda = (\lambda_1 \cdots \lambda_r)$ of n, with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r$, construct a frame of shape λ by placing λ_1 boxes in the first row, λ_2 boxes in the second row, and so on, with the first box from each row lying in the same vertical column. Fill in the boxes with the symbols $1,..., n$ in any way, to obtain a diagram D of shape λ. Let $R(D)$ be the subgroup of S_n consisting of all permutations that leave the set of symbols in each row invariant, and $C(D)$ the subgroup leaving the set of symbols in each column invariant. Then in $\mathbb{C}[S_n]$, define

$$P(D) = \sum_{r \in R(D)} r \qquad N(D) = \sum_{c \in C(D)} c\epsilon_c ,$$

where ϵ_c is the parity of the permutation c. Then define

$$e(D) = P(D)\, N(D) = \sum_{\substack{r \in R(D) \\ c \in C(D)}} rc\epsilon_c .$$

If $g \in S_n$, we may allow g to act on the symbols in the boxes of the frame, and gD will be a different diagram with the same shape λ. It is then immediate that $e(gD) = ge(D)\, g^{-1}$. Moreover, $e(D)$ is, up to a scalar factor, idempotent. It generates a minimal left ideal of $\mathbb{C}[S_n]$, on which $\mathbb{C}[S_n]$ acts with character χ^λ. Therefore

$$I_\lambda = (1/n!) \sum_{g \in S_n} ge(D)g^{-1}$$

is clearly central, in the simple component of $\mathbb{C}[S_n]$ corresponding to χ^λ, and is therefore a scalar multiple of χ^λ. Considering the coefficient of the identity element shows that

$$I_\lambda = \chi / f_\chi . \tag{7}$$

The standard proofs of these results make use of the following lemma, which will also be needed in this paper.

LEMMA 1. *$x \in \mathbb{C}[S_n]$ satisfies $pxq = \epsilon_q x$ for all $p \in R(D)$, and all $q \in C(D)$ if and only if $x = \gamma e(D)$ for some $\gamma \in \mathbb{C}$.*

We will also need the following.

LEMMA 2. *If* $s \in S_n$, *then* $e(D)\, se(D) = \gamma(s)\, e(D)$, *where* $\gamma(s) \in \mathbb{C}$ *is the coefficient of* s^{-1} *in* $[e(D)]^2$.

Proof. $e(D)\, se(D)$ satisfies the condition in Lemma 1. Therefore $e(D)\, se(D) = \gamma(s)\, e(D)$ for some $\gamma(s) \in \mathbb{C}$. Comparing coefficients of 1 we see that $\gamma(s)$ must be the coefficient of 1 in $e(D)\, se(D)$. Therefore

$$\gamma(s) = \sum_{pqsp'q'=1} \epsilon_q \epsilon_{q'} = \sum_{s^{-1}=p'q'pq} \epsilon_q \epsilon_{q'},$$

which is the coefficient of s^{-1} in $[e(D)]^2$.

We are now in a position to construct $q(n)$ mutually annihilating idempotents in $\mathbb{C}[S_n]^{S_{n-1}}$. Let $\underline{\lambda} = (\lambda_1 \cdots \underline{\lambda}_i \cdots \lambda_r)$ be a 1-partition of n. Assume the parts have been numbered in decreasing order, and that $\lambda_i > \lambda_{i+1}$ (that is, if the distinguished part happens to equal some undistinguished parts, the distinguished part comes last). Construct a frame of shape λ as before, and note that distinguishing the part λ_i is equivalent to distinguishing the corner at the end of the ith row. We insert the symbol 1 in this distinguished corner, and the result, as in Fig. 1, is called a frame of shape $\underline{\lambda}$.

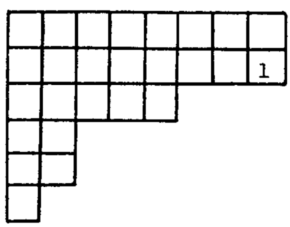


FIG. 1. A frame of shape $\underline{\lambda} = (8, \underline{8}, 5, 2, 2, 1)$.

Fill in the remaining boxes with the symbols $2, \ldots, n$ in any manner. Call the resulting diagram D, and define

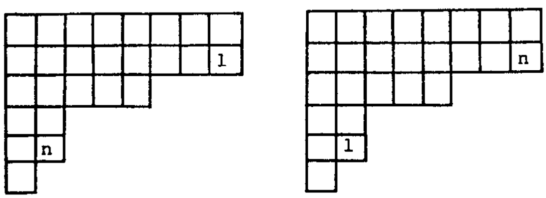
$$I_{\underline{\lambda}} = (1/(n-1)!) \sum_{h \in S_{n-1}} he(D)h^{-1}.$$

Note immediately that $I_{\underline{\lambda}}$ depends only on the 1-partition $\underline{\lambda}$. Filling in the symbols $2, \ldots, n$ differently only conjugates $e(D)$ by an element of S_{n-1} and therefore only permutes the summands. Second, since for the same reason conjugating $I_{\underline{\lambda}}$ by an element of S_{n-1} leaves it fixed, we have $I_{\underline{\lambda}} \in \mathbb{C}[S_n]^{S_{n-1}}$. Third, since the coefficient of the identity element in $e(D)$ is 1, the coefficient of the identity in $I_{\underline{\lambda}}$ is also 1, and therefore $I_{\underline{\lambda}} \neq 0$. Finally, since a semisimple commutative algebra can have no nilpotents, we know that $[I_{\underline{\lambda}}]^2 \neq 0$.

PROPOSITION 2. *If $\underline{\lambda}$ and $\underline{\mu}$ are any two distinct* 1-*partitions then* $I_{\underline{\lambda}} \cdot I_{\underline{\mu}} = 0$.

Proof. Case 1. χ and μ are already distinct as partitions. Then all summands in $I_{\underline{\lambda}}$ lie in the simple component of $\mathbb{C}[S_n]$ corresponding to the character $[\lambda]$, and those of $I_{\underline{\mu}}$ lie in the component corresponding to the character $[\mu]$. Hence every summand of $I_{\underline{\lambda}}$ annihilates every summand of $I_{\underline{\mu}}$.

Case 2. $\underline{\lambda}$ and $\underline{\mu}$ have the same underlying partition, but different distinguished parts. Since $\mathbb{C}[S_n]^{S_{n-1}}$ is commutative, we may assume that $\underline{\lambda}$ has a larger distinguished part than $\underline{\mu}$, reversing the roles of the two 1-partitions if necessary.



D D'

FIG. 2. Frames of shape $\underline{\lambda} = (8, \underline{8}, 5, 2, 2, 1)$ and of Shape $\underline{\mu} = (8, 8, 5, 2, \underline{2}, 1)$.

In Fig. 2, let D be a frame of shape $\underline{\lambda}$, and D' a frame of shape $\underline{\mu}$. Since the rest of the filling in is arbitrary, place the symbol n in each frame in the square occupied by 1 in the other, and fill in the remaining boxes identically. Let s be the transposition $(1n)$. Then $e(D') = se(D)\,s^{-1}$. Then

$$I_{\underline{\lambda}} = (1/(n-1)!) \sum_{h \in S_{n-1}} he(D)h^{-1} \qquad I_{\underline{\mu}} = (1/(n-1)!) \sum_{k \in S_{n-1}} kse(D)s^{-1}k^{-1}.$$

Hence

$$I_{\underline{\lambda}} \cdot I_{\underline{\mu}} = (1/(n-1)!^2) \sum_{h,k \in S_{n-1}} he(D)h^{-1}\, kse(D)s^{-1}k^{-1}.$$

Letting $h^{-1}k = t$,

$$I_{\underline{\lambda}} \cdot I_{\underline{\mu}} = (1/(n-1)!^2) \sum_{h,t \in S_{n-1}} he(D)\, tse(D)s^{-1}t^{-1}h^{-1}.$$

By Lemma 2, $e(D)\, tse(D) = e(D)\, \gamma(ts)$, so

$$I_{\underline{\lambda}} \cdot I_{\underline{\mu}} = (1/(n-1)!^2) \sum_{h,t \in S_{n-1}} he(D)\, \gamma(ts)s^{-1}t^{-1}h^{-1}$$

$$= (1/(n-1)!^2) \sum_{h \in S_{n-1}} he(D) \left[\sum_{t \in S_{n-1}} \gamma(ts)s^{-1}t^{-1}\right] h^{-1}.$$

The inner sum is merely all elements of S_n of the form $s^{-1}t^{-1}$ with $t \in S_{n-1}$, multiplied by their coefficients in $[e(D)]^2$. Since $e(D)$ is, up to a scalar, idempotent, let us consider the coefficient of $s^{-1}t^{-1}$ in $e(D)$. All terms in $e(D)$ are of the form pq with $p \in R(D)$ and $q \in C(D)$. Suppose then, that $s^{-1}t^{-1} = pq$. Since $t \in S_{n-1}$, $s^{-1}t^{-1}(1) = n$. Therefore $pq(1) = n$. But $q(1)$ must be some symbol in the column above 1 in diagram D, and p can only carry that symbol to something in that row. Hence $pq(1) = n$ is impossible. Therefore $\gamma(ts) = 0$ for all $t \in S_{n-1}$ and the inner sum is zero. Hence the result.

PROPOSITION 3. *The $I_{\underline{\lambda}}$'s are linearly independent.*

Proof. This is an immediate consequence of Proposition 2 and the fact that $[I_{\underline{\lambda}}]^2 \neq 0$.

PROPOSITION 4. *Up to scalars, the $I_{\underline{\lambda}}$'s are idempotent.*

Proof. The $I_{\underline{\lambda}}$'s lie in the vector space $\mathbb{C}[S_n]^{S_{n-1}}$ whose dimension over $\mathbb{C}$ is $q(n)$. By Proposition 3, the $I_{\underline{\lambda}}$'s therefore form a basis. Let

$$1 = \sum_{\underline{\lambda}} b_{\underline{\lambda}} I_{\underline{\lambda}},$$

for some scalars $b_{\underline{\lambda}} \in \mathbb{C}$. Multiplying by $I_{\underline{\mu}}$ yields

$$I_{\underline{\mu}} = b_{\underline{\mu}}[I_{\underline{\mu}}]^2.$$

Hence, up to scalar factors, the $I_{\underline{\lambda}}$'s are the spherical functions. In fact, the scalar factors are all unity, since the coefficient of 1 in $I_{\underline{\lambda}}$ is 1, and $Y_{\chi\psi}(1) = c_{\chi\psi} = 1$, since S_{n-1} is a multiplicity-free subgroup of S_n. The only question to settle is, Which $Y_{\chi\psi}$ is $I_{\underline{\lambda}}$ equal to? The answer, naturally enough, is

THEOREM 1. $I_{\underline{\lambda}} = [\underline{\lambda}]$.

Proof. Let χ' be any character of S_n other than $[\lambda]$. Then

$$I_{\underline{\lambda}} * \chi' = (1/n!(n-1)!) \sum_{h \in S_{n-1}} he(D)h^{-1}\chi'.$$

Since every summand in $I_{\underline{\lambda}}$ is in the simple component of $\mathbb{C}[S_n]$ corresponding to $[\lambda]$, they all annihilate χ', which by assumption lies in another component. By relation (2), there is one character that does not annihilate $Y_{\chi\psi}$, namely, χ. The only one left is $\chi = [\lambda]$.

Now let $\psi = [\lambda_1 \cdots (\lambda_i - 1) \cdots \lambda_r] \in \hat{S}_{n-1}$; let D be a diagram of shape $\underline{\lambda}$, and D' refer to the diagram D with the distinguished corner removed.

Thus D' contains only the symbols $2,\ldots, n$ and $e(D')$ is a multiple of a primitive idempotent associated to ψ. It is clear that $e(D')$ is equal to $e(D)$ with those terms involving elements not in S_{n-1} removed. That is,

$$e(D) = e(D') + \sum_{\substack{p \notin S_{n-1} \\ \text{or} \\ q \notin S_{n-1}}} pq\epsilon_q . \tag{8}$$

Then

$$I_\lambda * \tilde{\psi} = \frac{1}{n!}\left[\frac{1}{(n-1)!}\sum_{h \in S_{n-1}} he(D)h^{-1}\right]\left[\frac{nf_\psi}{(n-1)!}\sum_{k \in S_{n-1}} ke(D')k^{-1}\right],$$

which, by (8), may be rewritten as the sum

$$I_\lambda * \tilde{\psi} = \frac{f_\psi}{(n-1)!^3}\left[\sum_{h \in S_{n-1}} he(D')h^{-1}\right]\left[\sum_{k \in S_{n-1}} ke(D')k^{-1}\right] \tag{A}$$

$$+ \frac{f_\psi}{(n-1)!^3}\left[\sum_{h \in S_{n-1}} h\left(\sum_{\substack{p \notin S_{n-1} \\ \text{or} \\ q \notin S_{n-1}}} pq\epsilon_q\right)h^{-1}\right]\left[\sum_{k \in S_{n-1}} ke(D')k^{-1}\right]. \tag{B}$$

Clearly (A) is a nonzero element of $\mathbb{C}[S_{n-1}]$. Further, every summand in the last factor of (B) leaves the symbol 1 fixed, while every summand in the first factor of (B) moves the symbol 1 (as in the proof of Proposition 2). Hence none of the terms in (A) can be canceled by anything from (B) and we have

$$I_\lambda * \tilde{\psi} \neq 0.$$

But by relation (3) the only $\tilde{\psi}'$ that does not annihilate $Y_{\chi\psi}$ is $\tilde{\psi}$ itself. Therefore if $I_\lambda = Y_{\chi\psi}$, we have shown that

$$\chi = [\lambda_1 \cdots \lambda_r]$$

and

$$\psi = [\lambda_1 \cdots (\lambda_i - 1) \cdots \lambda_r]. \qquad \text{Q.E.D.}$$

4. The Averaging Theorem

Having established that $[\lambda] = (1/(n-1)!)\sum_{h \in S_{n-1}} he(D)\,h^{-1}$, we may interpret this as follows. The value of $[\lambda]$ on an element σ is the average of the coefficients in $e(D)$ over the S_{n-1}-class containing σ. We know from relation (7) that the average of the coefficients in $e(D)$ over the entire S_n-class containing σ gives the value of χ/f_χ at σ.

Denote the S_n-class containing σ by K, and suppose K breaks up into the S_{n-1}-classes $K_1, \ldots, K_t$. Then since the average of coefficients over all of K is just the average of the averages within each K_s weighted according to the sizes of the various K_s's, we have

$$\chi(K)/f_\chi = (1/\| K \|) \sum_{s=1}^{t} \| K_s \| \, Y_{\chi\psi}(K_s), \tag{9}$$

where $\chi(K)$ means χ evaluated at any element of K, and similarly for $Y_{\chi\psi}(K_s)$. Note that ψ can be any character of S_{n-1} contained in χ, as long as it is held fixed throughout the sum. In fact, with slight adjustment to account for multiplicities, (9) holds in the most general setting[1]:

THEOREM 2. *Let G be any finite group, H any subgroup. Choose any $\chi \in \hat{G}$, and any $\psi \in \hat{H}$. Let K be a conjugacy class of G, and $K_1, \ldots, K_t$ the H-conjugacy classes making up K. Then*

$$\frac{c_{\chi\psi}}{f_\chi} \| K \| \, \chi(K) = \sum_{\sigma \in K} Y_{\chi\psi}(\sigma) = \sum_{s=1}^{t} \| K_s \| \, Y_{\chi\psi}(K_s).$$

Proof. The second equality is clear. To prove the first, evaluate relation (2) at the identity, and obtain

$$(Y_{\chi\psi}, \chi') = \delta_{\chi\chi'} c_{\chi\psi}/f_\chi = (c_{\chi\psi}/f_\chi)(\chi, \chi').$$

Therefore $(Y_{\chi\psi},\ )$ and $(c_{\chi\psi}/f_\chi)(\chi,\ )$ are linear functionals on the space of class functions on G which agree on all of the characters of G. But the characters span the space of all class functions, and so these two functionals must agree on any class function. Let φ_K be the function which is 1 on any element of K, and 0 elsewhere. Then φ_K is certainly a class function, and so

$$(Y_{\chi\psi}, \varphi_K) = (c_{\chi\psi}/f_\chi)\,(\chi, \varphi_K),$$

$$(1/\| G \|) \sum_{\sigma \in G} Y_{\chi\psi}(\sigma) \overline{\varphi_K(\sigma)} = (c_{\chi\psi}/\| G \| f_\chi) \sum_{\sigma \in G} \chi(\sigma) \overline{\varphi_K(\sigma)},$$

$$\sum_{\sigma \in K} Y_{\chi\psi}(\sigma) = (c_{\chi\psi}/f_\chi) \sum_{\sigma \in K} \chi(\sigma) = (c_{\chi\psi}/f_\chi) \| K \| \, \chi(K). \qquad \text{Q.E.D.}$$

[1] The referee points out that Theorem 2 may also be viewed as a special case of a result on centralizer rings: The algebra $\mathbb{C}[G]^H$ is isomorphic to the centralizer ring for the permutation module associated with the action of $G \times H$ on G by left and right multiplication. If ξ is the character of an irreducible constituent of this module (e.g., $\xi = \chi\bar{\psi}$) and ζ is the corresponding character of the centralizer ring ($\zeta = Y_{\chi\psi}$) then we have $\xi(\underline{K})/\xi(1) = \zeta(\underline{K})/\zeta(1)$ for any class sum $\underline{K}$ (where K is a conjugacy class of $G \times H$) since both numbers give the scalar by which $\underline{K}$ acts on the ξ-component of the permutation module. Interpreted for $K \subseteq G$, this gives Theorem 2. Namely, if $K = L \times M$, we have $\chi(\underline{L})\bar{\psi}(\underline{M})/f_\chi f_\psi = Y_{\chi\psi}(\underline{L}\underline{M}^{-1})/c_{\chi\psi}$. Letting M be the unit class in H gives the result.

5. Calculation of Spherical Function Values for $S_{n-1} \subset S_n$

In principle, Theorem 1 allows the calculation of any value of $[\underline{\lambda}]$, although the operations entailed are quite lengthy. Nevertheless, it is faster than working directly from the definition given in (1). Certain values may be computed immediately. First, since S_{n-1} is a multiplicity-free subgroup of S_n, definition (1) gives $[\underline{\lambda}](1) = 1$ for all $\underline{\lambda}$. In a similar vein, since $[n]$ is the trivial character of S_n and $[n-1]$ the trivial character of S_{n-1}, $[\underline{n}]$ will be the trivial spherical function; i.e., $[\underline{n}](\sigma) = 1$ for all $\sigma \in S_n$.

Suppose next that a partition of n has only one distinct part, so that $\lambda_1 = \lambda_2 = \cdots = \lambda_r = a$. Denoting this partition by (a^r), it is clear that the character $[a^r]$ of S_n contains only one character of S_{n-1} on restriction, namely $[a^{r-1}, (a-1)]$. Then the spherical function $[\underline{a}^r]$ can be calculated directly. By (5),

$$[a^r] = f_{[a^{r-1},(a-1)]}[\underline{a}^r],$$

in which the right-hand side is the only summand indicated in (5). Since the degree of $[a^r]$ equals the degree of $[a^{r-1}, (a-1)]$, it follows that

$$[\underline{a}^r] = [a^r]/f_{[a^r]}.$$

In the same manner, for any finite group G and subgroup H, the values of any $Y_{\chi\psi}$ can be calculated directly from the character table of G whenever $\chi\,|_H = \psi$.

Now consider the S_{n-1}-class $(\underline{a}^r)$. It is clearly equal to the entire S_n-class (a^r). Therefore, by the averaging theorem, the value of any $Y_{\chi\psi}$ on $(\underline{a}^r)$ is just $c_{\chi\psi}\chi(a^r)/f_\chi$ or,

$$[\underline{\lambda}](\underline{a}^r) = [\lambda](a^r)/f_{[\lambda]}\,.$$

We can also calculate the value of any spherical function on an element of the subgroup. From definition (1) it follows that

$$Y_{\chi\psi}\,|_H = c_{\chi\psi}\psi/f_\psi\,.$$

The elements of S_{n-1} make up precisely those S_{n-1}-classes of S_n represented by 1-partitions having a distinguished part of size 1. Let $\underline{\mu} = (\mu_1, \ldots, \mu_s, \underline{1})$ be such a 1-partition. Then the above equation can be rewritten as

$$[\lambda_1 \cdots \underline{\lambda}_i \cdots \lambda_r](\underline{\mu}) = \frac{[\lambda_1 \cdots (\lambda_i - 1) \cdots \lambda_r](\mu_1 \cdots \mu_s)}{f_{[\lambda_1 \ldots (\lambda_i - 1) \ldots \lambda_r]}},$$

for any $\underline{\lambda}$, and can be read off the S_{n-1} character table.

Consider finally an S_n-class K with two distinct parts, one of which is 1. So $K = (a^r, 1^s)$. Then K breaks up into two S_{n-1}-classes, $(a^r, 1^s) = (\underline{a}^r, 1^s) \cup (a^r, \underline{1}^s)$. By the above remark, we know the value of every spherical function on the latter class. We can then use the averaging theorem to calculate the values on the other class.

In order to use the averaging theorem, one has to know the size of a given S_{n-1}-class. Let $\underline{\lambda} = (\lambda_1 \cdots \underline{\lambda}_i \cdots \lambda_r)$ be a 1-partition. If α of the parts are equal to 1, β of the parts are equal to 2, and so on, and if δ of the parts are equal to the distinguished part λ_i, then rewrite the 1-partition as $(1^\alpha 2^\beta \cdots \underline{\lambda}_i{}^\delta \cdots)$. A simple counting argument then shows that

$$\|(1^\alpha 2^\beta \cdots \underline{\lambda}_i{}^\delta \cdots)\| = \frac{(n-1)!}{1^\alpha \alpha!\, 2^\beta \beta! \cdots \lambda_i^{\delta-1}(\delta-1)! \cdots}.$$

6. Conjugate Spherical Functions

There is a well-known involution on the set of partitions of n called conjugation. If λ is a partition, draw a frame of shape λ, and then reflect it in the main diagonal. The shape of the resulting frame is the conjugate partition, denoted λ^*. The corresponding involution on the elements of $\hat{S}_n$, namely, $[\lambda] \mapsto [\lambda^*]$ is just the action on $\hat{S}_n$ of the nontrivial one-dimensional character $[1^n]$, whose value on a permutation σ is ϵ_σ. In other words, $[\lambda^*](\sigma) = \epsilon_\sigma[\lambda](\sigma)$.

Spherical functions can be conjugated in an analogous way. Let $\underline{\lambda}$ be a 1-partition of n. Draw a frame of shape $\underline{\lambda}$, and shade the box in the distinguished corner. Reflect the frame in the main diagonal, reflecting the shading along with the frame. The 1-partition corresponding to the reflected frame we call $\underline{\lambda}^*$, the conjugate 1-partition. It is obvious that $\underline{\lambda}^{**} = \underline{\lambda}$, and that if

$$\underline{\lambda} = (\lambda_1 \cdots \underline{\lambda}_i \cdots \lambda_r)$$

and

$$\underline{\lambda}^* = (\mu_1 \cdots \underline{\mu}_j \cdots \mu_s),$$

then

$$\lambda^* = (\mu_1 \cdots \mu_s)$$

and

$$(\lambda_1 \cdots (\lambda_i - 1) \cdots \lambda_r)^* = (\mu_1 \cdots (\mu_j - 1) \cdots \mu_s).$$

We can then define the conjugate to the spherical function $[\underline{\lambda}]$ to be the spherical function $[\underline{\lambda}^*]$. We will then have $[\underline{\lambda}^*](\sigma) = \epsilon_\sigma[\underline{\lambda}](\sigma)$. These remarks constitute a special case of the following more general situation.

Let G be a finite group, and H a subgroup. Let $\hat{G}_0$ be the group of one-dimensional characters of G. Then the action of $\hat{G}_0$ on $\hat{G}$ and $\hat{H}$ by multiplication induces an action of $\hat{G}_0$ on the spherical functions for $H \subset G$ in the following way. For $\chi \in \hat{G}$, $\psi \in \hat{H}$, and $\eta \in \hat{G}_0$, define

$$(\eta Y_{\chi\psi}) = Y_{\eta\chi,\eta\psi}.$$

Then we have

THEOREM 3. $(\eta Y_{\chi\psi})(\sigma) = \eta(\sigma)\, Y_{\chi\psi}(\sigma)$.

Proof.

$$\begin{aligned} Y_{\eta\chi,\eta\psi}(\sigma) &= (1/\| H \|) \sum_{\tau \in H} \eta\chi(\sigma\tau)\, \eta\psi(\tau^{-1}) \\ &= (1/\| H \|) \sum_{\tau \in H} \eta(\sigma\tau)\, \chi(\sigma\tau)\, \eta(\tau^{-1})\, \psi(\tau^{-1}) \\ &= \eta(\sigma)\, (1/\| H \|) \sum_{\tau \in H} \chi(\sigma\tau)\, \psi(\tau^{-1}) = \eta(\sigma)\, Y_{\chi\psi}(\sigma). \end{aligned}$$

In particular, note that

$$c_{\eta\chi,\eta\psi} = Y_{\eta\chi,\eta\psi}(1) = \eta Y_{\chi\psi}(1) = \eta(1)\, Y_{\chi\psi} = c_{\chi\psi}.$$

In the setting $S_{n-1} \subset S_n$, it may happen that $[\underline{\lambda}^*] = [\underline{\lambda}]$. As a consequence, $[\underline{\lambda}]$ will have to vanish on all odd classes. It is not difficult to see that $\underline{\lambda}$ is a self-conjugate 1-partition if and only if λ is a self-conjugate partition *and* the distinguished corner lies on the main diagonal.

There is a well-known method of setting up a bijection between self-conjugate partitions of n and partitions of n into distinct, odd-sized parts. One counts the boxes lying in the first row together with the first column to obtain the first odd part. Next one counts the boxes in the remainder of the second row together with the remainder of the second column, and so on. Conversely, given a partition of n into distinct, odd-sized parts, one can reconstruct a self-conjugate partition of n (see, for example [2]).

In the same way, one obtains a bijection between the self-conjugate 1-partitions of n and the partitions of n into distinct, odd-sized parts, one of which is of size 1. There are no self-conjugate 1-partitions for $n = 2, 3, 5$, or 7, but every other n has at least one. To see this, note that for $n = 1, 4$, and 6 we have the 1-partitions $(\underline{1})$, $(\underline{2}^2)$, and $(3, \underline{2}, 1)$. If $n > 7$ and n is even, then the partition $n = 1 + (n - 1)$ yields, by the method indicated above, a self-conjugate 1-partition. If $n > 7$ and n is odd, we have the partition $n = 1 + 3 + (n - 4)$.

A generating function for the number of self-conjugate 1-partitions of n is then clearly

$$J(x) = x(1 + x^3)(1 + x^5)(1 + x^7) \cdots.$$

For comparison, a generating function for $q(n)$, the total number of 1-partitions of n is

$$H(x) = \frac{x + x^2 + x^3 + \cdots}{(1 - x)(1 - x^2)(1 - x^3) \cdots},$$

which follows directly from the generating function for $p(n)$ and from Proposition 1.

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